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On Multiple Phase Transitions for Branching Markov Chains

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We consider branching Markov chains on a countable set. We give a necessary and sufficient condition in terms of the transition kernel of the underlying Markov chain to have two phase transitions. We compute the critical values. We apply this result to prove that asymmetric branching random walks on Z have two phase transitions.

KEY WORDS: Branching Markov chain; branching random walk; phase transition.

1. THE MODEL AND THE RESULTS

Let $p(x, y)$ be the transition kernel of a given Markov chain on a countable set S. The evolution of a branching Markov chain on S is governed by the following rules. A particle at x gives birth to a new particle at y at rate $\lambda p(x, y)$, where $\lambda > 0$ is a parameter. A particle dies at rate 1.

We will also be interested in the contact process, which is a Markov process with the same birth and death rates, but for which we do not allow more than one particle per site. For the contact process $p(x, y) = 0$ unless x and y are "nearest neighbors" with respect to a given distance on S.

Let η_t^x denote the branching Markov chain starting from a single particle at x and let $\eta_t^x(y)$ be the number of particles at site y at time t. We denote the total number of particles of η_t^x by $|\eta_t^x| = \sum_{y \in S} \eta_t^x(y)$. Let O be a fixed site of S. We define the following critical parameters:

$$\lambda_1 = \inf\{\lambda: P(|\eta_t^O| \geqslant 1, \forall t > 0) > 0\}$$

$$\lambda_2 = \inf\{\lambda: P(\limsup_{t \to \infty} \eta_t^O(O) \geqslant 1) > 0\}$$

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The process $|\eta_t^O|$ is a (nonspatial) Galton–Watson process, so $\lambda_1 = 1$ for any $p(x, y)$ and any S.

We will say that we have two phase transitions when $\lambda_1 < \lambda_2$ (note that λ_1 is always smaller than or equal to λ_2). Our purpose here is to give a necessary and sufficient condition to have two phase transitions and to give an explicit formula for λ_2 depending on $p(x, y)$. To do so, we need to analyze $P_t(x, y)$, the continuous-time Markov chain corresponding to $p(x, y)$. More precisely, the evolution of $P_t(x, y)$ is governed by the following rule; after a mean 1 exponential time it jumps, going from x to y with probability $p(x, y)$. By the Markov property we have

$$P_{t+s}(O, O) \geqslant P_t(O, O)\, P_s(O, O)$$

This implies the existence of the following limit:

$$\lim_{t \to \infty} \frac{1}{t} \log P_t(O, O) = -\gamma = \sup_{t > 0} \frac{1}{t} \log P_t(O, O) \tag{1}$$

Moreover, it is clear that $\gamma \geqslant 0$ and since $P_t(O, O) \geqslant e^{-t}$ (if there are no jumps up to time t), we get that γ is in $[0, 1]$.

We are now ready to state our result.

Theorem 1. The second critical value for a branching Markov chain is

$$\lambda_2 = \frac{1}{1-\gamma} \qquad \text{for} \quad \gamma \text{ in } [0, 1)$$

$$\lambda_2 = \infty \qquad \text{for} \quad \gamma = 1$$

In particular, there are two phase transitions for this model if and only if $\gamma \neq 0$.

The problem of the existence of two phase transitions has been considered by Pemantle,[5] who proved that the symmetric contact process on a homogeneous tree has two phase transitions if each site of the tree has four neighbors or more, and by Schonmann,[6] who proved that the asymmetric contact process on Z has two phase transitions if there is enough asymmetry. The symmetric contact process on Z^d has been widely studied[1-3] and it is known that for this process $\lambda_1 = \lambda_2$.

Our motivation for this work was to understand the appearance of two phase transitions; we were in particular interested in understanding if the two phase transitions for the contact process on a tree and for the asymmetric contact process on Z had the same cause. While we were not able to answer these questions for the contact process, Theorem 1 provides a very simple necessary and sufficient condition to have two phase transitions for any branching Markov chain on any countable set. This makes us conjecture that $\gamma \neq 0$ is also a necessary and sufficient condition for the contact process to have two phase transitions.

Madras and Schinazi[4] analyzed several models and computed λ_2 for a symmetric branching random walk on a homogeneous tree, showing that $\lambda_1 < \lambda_2$ on any homogeneous tree for which each site has three neighbors or more. The computation there uses in a crucial way the symmetry of the model. In particular, it is not possible to adapt the argument in ref. 4 to treat an asymmetric branching random walk on Z, and a new argument to prove Theorem 1 is needed. The new argument used here is simpler (it involves only a first moment analysis) and applies to a much more general situation.

In ref. 4 there is also a proof that when there are two phase transitions, the second phase transition is not continuous in the following sense.

Corollary 1. If γ is in $(0, 1)$, then the function $\lambda \to P(\limsup_{t\to\infty} \eta_t^O(O) \geqslant 1)$ is not continuous at λ_2.

The proof in ref. 4 (Theorem 4) works here, too, without modification.

2. PROOF OF THEOREM 1

The key to our analysis is the following lemma.

Lemma 1. If there is a time T such that $E(\eta_T^O(O)) > 1$, then $\limsup_{t\to\infty} P(\eta_t^O(O) \geqslant 1) > 0$.

Proof of Lemma 1. We will construct a supercritical Galton–Watson process which is dominated (in a certain sense) by the branching Markov chain. To do so, we will first consider the following Markov process $\tilde{\eta}_t$ whose evolution is coupled with the evolution of η_t^O in the following way. Up to time T, $\tilde{\eta}_t$ and η_t^O are identical. At time T we suppress all the particles of $\tilde{\eta}_t$ which are not at site O and we keep the particles which are at O. Between times T and $2T$ the particles of $\tilde{\eta}_t$ which were at O at time T evolve like the particles of η_t^O which were at O at time T. At time $2T$ we suppress again all the particles of $\tilde{\eta}_t$ which are not at O. And so on, at times kT $(k \geqslant 1)$ we suppress all the particles of $\tilde{\eta}_t$ which are not at O and between kT and $(k+1)T$ the particles of $\tilde{\eta}_t$ evolve like the corresponding particles of η_t^O.

Now we can define the following discrete-time process ξ_i. Let $\xi_0 = 1$ and $\xi_i = \tilde{\eta}_{iT}(O)$. Using the fact that each particle evolves independently of the other particles, it is clear that ξ_i is a Galton–Watson process. Moreover, $E(\xi_1) > 1$, so ξ_i is a supercritical Galton–Watson process. On the other hand, by our construction $\tilde{\eta}_t(x) \leqslant \eta_t^O(x)$ for all x in S and all $t \geqslant 0$. And so

$$P(\xi_i \geqslant 1) \leqslant P(\eta_{iT}^O(O) \geqslant 1) \tag{2}$$

But $P(\xi_i \geqslant 1, \forall i \geqslant 0) > 0$, so making i go to infinity in (2) concludes the proof of Lemma 1.

It is now easy to prove Theorem 1. Let $m_t(x) = E(\eta_t^x(O))$ be the expected number of particles at O. We have the following representation of $m_t(x)$ [see, for instance, (2.3) in ref. 4]

$$m_t(x) = e^{(\lambda - 1)t} P_{\lambda t}(x, O) \tag{3}$$

Using (1) and (3), we get for all $t \geqslant 0$

$$m_t(O) \leqslant e^{Ct} \tag{4}$$

for $C = \lambda - 1 - \lambda\gamma$. If $\gamma < 1$, we observe that if $\lambda < 1/(1-\gamma)$, then $C < 0$. But $P(\eta_t^O(O) \geqslant 1) \leqslant m_t(O)$, so using the Borel–Cantelli lemma, we find that (4) implies that

$$P(\limsup_{t \to \infty} \eta_t^O(O) \geqslant 1) = 0$$

and this shows that $\lambda_2 \geqslant 1/(1-\gamma)$ for $\gamma < 1$. For $\gamma = 1$, (4) holds with $C = -1$, so $\lambda_2 = \infty$.

For $\gamma < 1$ we will now prove the other inequality. Suppose that $\lambda > 1/(1-\gamma)$. Again, using (1) and (3), we get that for t sufficiently large there is a constant $D > 0$ depending on λ and γ such that

$$m_t(O) \geqslant e^{Dt}$$

But now we can apply Lemma 1 and so we get that $\lambda_2 \leqslant 1/(1-\gamma)$. This finishes the proof of Theorem 1.

3. ONE APPLICATION: THE ASYMMETRIC BRANCHING RANDOM WALK

Consider the asymmetric branching random walk (ABRW) on Z. A particle at x gives birth to a particle at $x+1$ at rate λ_r. A particle at x gives birth to a particle at $x-1$ at rate λ_l. A particle dies at rate 1. So $\lambda = \lambda_r + \lambda_l$, $p(x, x+1) = \lambda_r/\lambda$ and $p(x, x-1) = \lambda_l/\lambda$. The other entries of $p(x, y)$ are zero.

For this example γ is easily computed. Let $p_{2n}(O, O)$ be the probability that the discrete-time random walk is in O at time $2n$. An elementary computation shows that

$$\lim_{n \to \infty} \frac{p_{2n}(O, O)}{(4pq)^n (\pi n)^{-1/2}} = 1$$

where $p=p(x,x+1)$ and $q=p(x,x-1)$. It is then easy to get the following limit in continuous time:

$$\lim_{t\to\infty}\frac{1}{t}\log P_t(O,O)=2(pq)^{1/2}-1=2\frac{(\lambda_r\lambda_l)^{1/2}}{\lambda}-1=-\gamma$$

The two critical values $\lambda_1=1$ and $\lambda_2=1/(1-\gamma)$ can be written using λ_r and λ_l. We get for the first critical line

$$\lambda_r+\lambda_l=1$$

and for the second critical line

$$4\lambda_r\lambda_l=1$$

In particular, the only intersection point of these two lines is $\lambda_r=\lambda_l=1/2$. This proves that any asymmetry in this model causes two phase transitions. This was conjectured by Schonmann[(6)] (see ref. 6, and open problem 2) for the contact process. The contact process has the same birth rates and death rates as the ABRW except that there is at most one particle per site (so if a birth is attempted at an occupied site, this birth is suppressed). This additional condition makes things much more delicate to analyze than for the ABRW. Schonmann was able to prove that there are two phase transitions for λ_r large enough, but the problem of showing that any asymmetry causes two phase transitions is still open. Our analysis for the ABRW supports his conjecture.

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